Announcements

1) Final 8-11 CB 1042
(usual classroom) Monday

Will have polar decomposition,
matrix norm

2) Look over old exams for
study aids - practice problems
on CTools under "Assignments"

3) Review 1-2:30
CB 2070

Recall:

best fit quadratic

Given $(x_1, y_1), \ldots, (x_n, y_n)$ in $\mathbb{R}^2$,

we find the best fit quadratic

$$ax^2 + bx + c = 0$$

as least-squares solutions to

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Example 1: (best fit quadratic)

Points

$(0,1), (1,1), (2,3), (3,2)$

Find the best-fit quadratic for these points.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{A} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

These are the same as actual solutions to

$$A^t A \begin{bmatrix} c \\ b \\ a \end{bmatrix} = A^t \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix}$$

$$\det(A^tA) = 80 \neq 0,$$

so invertible.

$$A^t \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \\ 31 \end{bmatrix}$$

Then

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} = (A^t A)^{-1} \begin{bmatrix} 7 \\ 13 \\ 31 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 19 & -21 & 5 \\ -21 & 49 & -15 \\ 5 & -15 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 13 \\ 31 \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 \\ 5/4 \\ -1/4 \end{bmatrix}$$

Quadratic: $-1/4\, x^2 + 5/4\, x + 3/4$

Gram - Schmidt

(Section 6.4)

Not on final!

Suppose $V \subseteq \mathbb{R}^n$ is a subspace. Suppose $\dim(V) = k$. Then we know V has a basis $\{v_1, v_2, \ldots, v_k\}$.

Gram-Schmidt procedure creates an orthogonal basis $\{u_1, u_2, \ldots, u_k\}$ as follows:

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{u_1 \cdot v_2}{\|u_1\|_2^2} u_1$$

check: $u_1 \cdot u_2 = 0$.

$$u_3 = v_3 - \frac{u_2 \cdot v_3}{\|u_2\|_2^2} u_2 - \frac{u_1 \cdot v_3}{\|u_1\|_2^2} u_1$$

The i^{th} basis vector

u_i, $2 \le i \le k$, is given by

$$\boxed{u_i = v_i - \sum_{j=1}^{i-1} \frac{u_j \cdot v_i}{\|u_j\|_2^2} u_j}$$

You show orthogonality

using mathematical induction.

Example 2: Orthogonalize

the vectors

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ 56 \end{bmatrix}}_{v_3}$$

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = v_2 - \frac{u_1 \cdot v_2}{\| u_1 \|_2^2} u_1$$

$$u_2 = v_2 - \frac{u_1 \cdot v_2}{\|u_1\|_2^2} u_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ -1/2 \\ 5 \end{bmatrix}$$

u_3

$$= v_3 - \frac{u_2 \cdot v_3}{\|u_2\|_2^2} u_2 - \frac{u_1 \cdot v_3}{\|u_1\|_2^2} u_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 56 \end{bmatrix} - \frac{\begin{bmatrix} 1/2 \\ -1/2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 56 \end{bmatrix}}{\begin{bmatrix} 1/2 \\ -1/2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/2 \\ 5 \end{bmatrix}} \begin{bmatrix} 1/2 \\ -1/2 \\ 5 \end{bmatrix}$$

$$- \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 56 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

= some vector

Given a subspace V of $\mathbb{R}^n$, how do we know there is an orthogonal projection onto V?

Take a basis for V, ortho gonalize using Gram-Schmidt. We can the define P onto V as a linear transformation using the orthogonal basis.